Topology and its Applications 16 (1983) 287–301
North-Holland

STRATIFIABLE SPACES DEFINED BY PAIR COLLECTIONS

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Received 21 September 1982
Revised 15 March 1983

We define a pair (F, U) to be a closed set F and an open set U such that $F \subset U$. A sequence of pair collections is used to characterize stratifiable spaces instead of a sequence of neighbornets. We introduce a new class of spaces, called regularly stratifiable spaces, which is defined in terms of pair collections. Every stratifiable μ-space is regularly stratifiable, and every regularly stratifiable space has a σ-almost locally finite base, thus is hereditary M_1. J. Nagata's problem for the dimension of M_1-spaces is answered positively in the class of regularly stratifiable spaces.

AMS Subj. Class.: 54E20, 54F45

stratifiable	regularly stratifiable	almost locally finite
M_1-space	μ-space	dimension

1. Introduction

The M_i-spaces, $i = 1, 2, 3$, were defined in 1961 by Ceder [4] as natural generalization of metrizable spaces. C. Borges renamed M_3-spaces "stratifiable spaces" and proved several important results concerning these spaces in [3]. The equivalence of M_3 and M_2 was proved independently by Gruenhage [5] and Junnila [11].

There are several different ways to define stratifiable spaces. A regular space X is stratifiable if and only if X has any of the following properties:

(a) X has a σ-cushioned pair base, [4].

(b) There is a stratification of X, [3].

(c) For each $x \in X$ there is a sequence $\langle U_n(x)\rangle$ of open neighborhoods of x such that if F is closed and $y \notin F$, then $y \notin \mathrm{Cl}(\bigcup\{U_n(x): x \in F\})$ for some n, [7].

An assignment of neighborhoods to the point of X is called a neighbornet [11]. So the above characterization (c) implies that stratifiable spaces are characterized by a sequence of neighbornets.

In this paper, we introduce the notion of a pair to give characterizations of stratifiable spaces. A pair (F, U) is a closed set F and an open set U such that $F \subset U$. A pair collection is in a sense a generalization of a neighbornet. For instance, if U is a neighbornet of a T_1-space X, then $\{(\{x\}, U(x)): x \in X\}$ is a pair collection.

The notion of a pair collection becomes interesting and useful if we add various assumptions to it. In fact the main purpose of this paper is to define a new class of stratifiable spaces, called *regularly stratifiable spaces*, which is defined in terms of a sequence of pair collections with a certain assumption of locally finite type.

The classic problem whether $M_3 \Rightarrow M_1$ remains open. But some partial results are known. Gruenhage [6] showed that every stratifiable F_σ-metrizable space is M_1. More generally Mizokami [15] noted that every stratifiable μ-space, which is a stratifiable space embedded in a countable product of paracompact F_σ-metrizable space, can be shown to be M_1 by the same technique as that of Gruenhage.

On the other hand, Itō and the author [10] introduced the notion of almost local finiteness as a generalization of local finiteness, and investigated the class of all spaces with a σ-almost locally finite base. The class is countably productive and hereditary, and the class contains every free L-space defined by Nagami [18] (especially every Lašnev space) and every space with a σ-closure preserving base consisting of clopen sets which Heath and Junnila [8] call "an M_0-space". Every closed image of a space with a σ-almost locally finite base is M_1. It is shown in [9] that every Ξ-product of spaces with a σ-almost locally finite base also has a σ-almost locally finite base.

Here we show that every stratifiable μ-space is regularly stratifiable and every regularly stratifiable space has a σ-almost locally finite base, thus is hereditarily M_1. This extends the recent result of M. Itō that every stratifiable F_σ-metrizable space with countable dimension has a σ-almost locally finite base.

These implications can be summarized in a diagram as follows:

$$\begin{tikzcd}
& \text{metric space} \arrow[dl] \arrow[dr] & \\
\text{La\v{s}nev space} \arrow[d] & & \text{stratifiable } F_\sigma\text{-metrizable space} \arrow[ddl] \\
\text{free } L\text{-space} \arrow[dr] & & \\
& \text{stratifiable } \mu\text{-space} \arrow[d, Rightarrow, "(1)"'] & \\
& \text{regularly stratifiable space} \arrow[d, Rightarrow, "(2)"'] & \\
& \text{space with a } \sigma\text{-almost locally finite base} \arrow[d, Rightarrow, "(3)"'] & \\
& M_1\text{-space} \arrow[d, Rightarrow, "(4)"'] & \\
& \text{stratifiable space} &
\end{tikzcd}$$

We do not know if the reverse implications of (1), (2), (3), and (4) of the above diagram are true. Indeed we do not know of a stratifiable space which is not a μ-space.

Dimension theoretical results are obtained. In particular we show that if a space X is regularly stratifiable, then $\operatorname{Ind} X \leq n$ if and only if X has a σ-closure preserving base $\mathscr{B}$ such that $\operatorname{Ind}(\partial B) \leq n-1$ for every $B \in \mathscr{B}$. Nagata [20] asked if this result is true for any M_1-space. So our result is a partial answer to his question, and generalize the result [6] for stratifiable F_σ-metrizable spaces and the result [15] for stratifiable μ-spaces.

2. Definition of pair collections and preliminary results

Throughout this paper, all spaces are assumed to be regular T_1. The letter N denotes the set of positive integers.

Definition 2.1. (1) We define a *pair* (F, U) of a space X to be a closed set F and an open set U such that $F \subset U$. A *pair collection* of a space X is a collection of pairs of X.

(2) *Let* $\mathscr{P}$ be a pair collection of a space X. Define

$$\mathscr{P}^F = \{F : (F, U) \in \mathscr{P} \text{ for some open set } U\},$$

$$\mathscr{P}^U = \{U : (F, U) \in \mathscr{P} \text{ for some closed set } F\}.$$

(3) A pair collection $\mathscr{P}$ of a space X is a *cover* if the closed collection $\mathscr{P}^F$ is a cover of X. $\mathscr{P}$ is *locally finite* (*σ-locally finite, point finite*) if the open collection $\mathscr{P}^U$ is locally finite (σ-locally finite, point finite).

(4) Let $\mathscr{P}$ be a pair collection of a space X and $A \subset X$. Define

$$\mathscr{S}(A, \mathscr{P}) = \{U : (F, U) \in \mathscr{P} \text{ for some closed set } F \text{ with } F \cap A \neq \emptyset\},$$

$$\begin{aligned}\mathscr{A}(A, \mathscr{P}) &= \mathscr{S}(X \backslash A, \mathscr{P}) \\ &= \{U : (F, U) \in \mathscr{P} \text{ for some closed set } F \text{ with } F \backslash A \neq \phi\}.\end{aligned}$$

We denote by $\mathrm{S}(A, \mathscr{P})$ and $\mathrm{A}(A, \mathscr{P})$ the sets $\bigcup \mathscr{S}(A, \mathscr{P})$ and $\bigcup \mathscr{A}(A, \mathscr{P})$.

(5) Let $\mathscr{P}$ and $\mathscr{Q}$ be pair collections of a space X. $\mathscr{Q}$ *refines* $\mathscr{P}$ if for each $(H, V) \in \mathscr{Q}$, there is $(F, U) \in \mathscr{P}$ such that $H \subset F$ and $V \subset U$. $\mathscr{Q}$ *weakly refines* $\mathscr{P}$ if $\mathrm{S}(\{x\}, \mathscr{Q}) \subset \mathrm{S}(\{x\}, \mathscr{P})$ for every $x \in X$.

Now, we are going to establish lemmas to obtain σ-discreteness and point finiteness of pair collections. The following lemma due to H. Junnila is the key to obtain σ-discreteness of pair collections.

Lemma 2.2 [11]. *Let U be an unsymmetric neighbornet of a semi-stratifiable space*

X. Then there exists a σ-discrete and closed cover $\mathscr{F}$ of X such that for each $F \in \mathscr{F}$, we have $U(x) = U(y)$ for every $x, y \in F$.

Lemma 2.3. *Let X be a collectionwise normal semi-stratifiable space. Suppose $\mathscr{P} = \{(\{x\}, U(x)): x \in X\}$ is a pair collection of X, where the neighbornet U is unsymmetric . Then there is a σ-discrete pair cover $\mathscr{Q}$ which weakly refines $\mathscr{P}$.*

Proof. By Lemma 2.2, we have a closed cover $\mathscr{F} = \bigcup\{\mathscr{F}_n: n \in N\}$ where each $\mathscr{F}_n$ is discrete and for each $F \in \mathscr{F}$, there is an open set $U(F)$ such that $U(x) = U(F)$ for every $x \in F$. Using the collectionwise normality of X, for every $n \in N$, there is a discrete open collection $\{V(F): F \in \mathscr{F}_n\}$ such that $F \subset V(F) \subset U(F)$ for each $F \in \mathscr{F}_n$. Put $\mathscr{Q} = \{(F, V(F)): F \in \mathscr{F}\}$. Observe that if $x \in F$, then $V(F) \subset U(F) = U(x)$. Hence for each $x \in X$, $\mathrm{S}(\{x\}, \mathscr{Q}) \subset U(x) = \mathrm{S}(\{x\}, \mathscr{P})$. Thus $\mathscr{Q}$ is a σ-discrete pair cover which weakly refines $\mathscr{P}$. □

Lemma 2.4. *Let $\mathscr{P}$ be a σ-discrete pair collection of a perfectly normal space X. Suppose $\bigcup \mathscr{P}^F$ is a G_δ-set. Then there is a σ-discrete and point finite pair collection $\mathscr{Q}$ such that*

(a) $\bigcup \mathscr{P}^F = \bigcup \mathscr{Q}^F$; and

(b) $\mathscr{Q}$ refines $\mathscr{P}$.

Proof. Let $\mathscr{P} = \bigcup\{\mathscr{P}_n: n \in N\}$ where each $\mathscr{P}_n$ is a discrete pair collection. Since $\bigcup \mathscr{P}^F$ is a G_δ-set, there is a decreasing sequence $\{W_i\}$ of open sets such that $\bigcup \mathscr{P}^F = \bigcap\{W_i: i \in N\}$. Put $F_n = \bigcup \mathscr{P}^F_n$ for each $n \in N$ and $F_0 = \emptyset$. Note that the perfect normality of X implies that each $X \setminus \bigcup\{F_i: i = 0, 1, \ldots, n-1\}$ is a cozero set. Therefore, for each $n \in N$, we can choose a point finite open collection $\{U_{n,k}: k \in N\}$ and a closed collection $\{F_{n,k}: k \in N\}$ such that $F_{n,k} \subset U_{n,k}$ for each $k \in N$ and $\bigcup\{U_{n,k}: k \in N\} = \bigcup\{F_{n,k}: k \in N\} = X \setminus \bigcup\{F_i: i = 0, 1, \ldots, n-1\}$. Define $V_{n,k} = U_{n,k} \cap W_{n+k}$, and $H_{n,k} = F_{n,k} \cap F_n$. Then for each $n \in N$, the sequence $\{V_{n,k}: k \in N\}$ of open sets and the sequence $\{H_{n,k}: k \in N\}$ satisfy:

(1) $H_{n,k} \subset V_{n,k} \subset W_{n+k}$ for each $k \in N$;

(2) $(\bigcup\{V_{n,k}: k \in N\}) \cap (\bigcup\{F_i: i = 0, 1, \ldots, n-1\}) = \emptyset$;

(3) $F_n \setminus \bigcup\{F_i: i = 0, 1, \ldots, n-1\} = \bigcup\{H_{n,k}: k \in N\}$; and

(4) $\{V_{n,k}: k \in N\}$ is point finite.

Now define

$$\mathscr{Q}_{n,k} = \{(F \cap H_{n,k}, U \cap V_{n,k}): (F, U) \in \mathscr{P}_n\}.$$

We show that $\mathscr{Q} = \bigcup\{\mathscr{Q}_{n,k}: n, k \in N\}$ is the desired pair collection. Since each $\mathscr{P}_n$ is discrete, clearly each $\mathscr{Q}_{n,k}$ is discrete. Hence $\mathscr{Q}$ is σ-discrete. It is easy to check condition (a) and (b). We need only show that $\mathscr{Q}$ is point finite.

Let $x \in X$. If $x \notin \mathscr{P}^F$, then x is contained in finitely many elements of $\{W_i: i \in N\}$. Hence by (1), x is contained in only finitely many elements of $\{V_{n,k}: n, k \in N\}$. Since each $\mathscr{Q}_{n,k}$ is discrete and $\bigcup \mathscr{Q}^U_{n,k} \subset V_{n,k}$, $\mathscr{Q}^U$ is point finite at x.

If $x \in \mathcal{P}^F$, then $x \in F_m$ for some m. Then by (2), $x \notin \bigcup\{V_{n,k}: n \geq m+1$ and $k \in N\}$. So x is only contained in elements of $\{V_{n,k}: n = 1, 2, \ldots, m$ and $k \in N\}$. By (4), for each n, the collection $\{V_{n,k}: k \in N\}$ is point finite. Thus x is contained in finitely many elements of $\{V_{n,k}: n, k \in N\}$, which completes the proof. □

Lemma 2.5. *Let X be a collectionwise normal semi-stratifiable space and $\mathcal{P} = \{(\{x\}, U(x)): x \in X\}$ is a pair cover of X where the neighbornet U is unsymmetric. Then there is a σ-discrete and point finite pair cover $\mathcal{Q}$ which weakly refines $\mathcal{P}$.*

Proof. Note that every normal semi-stratifiable space is perfectly normal. Apply Lemma 2.4 after using Lemma 2.3. □

Lemma 2.6. *Let $\mathcal{P}$ be a σ-discrete pair cover of a perfectly normal space X, and let A and B be a closed sets of X. Then there is a σ-discrete and point finite pair collection $\mathcal{Q}$ such that*

(a) $\bigcup \mathcal{Q}^F = A \setminus B$, $\bigcup \mathcal{Q}^U \cap B = \emptyset$; *and*

(b) *$\mathcal{Q}$ refines $\mathcal{P}$.*

Proof. Let $\mathcal{P} = \bigcup\{\mathcal{P}_n: n \in N\}$ where each $\mathcal{P}_n$ is a discrete pair collection. Note that the perfect normality of X implies that $A \setminus B$ is F_σ and G_δ in X. Suppose $A \setminus B = \bigcup\{K_i: i \in N\}$ where each K_i is closed in X. Define

$$\mathcal{P}_{n,i} = \{(F \cap K_i, U): (F, U) \in \mathcal{P}_n\}.$$

Then each $\mathcal{P}_{n,i}$ is a discrete pair collection. Put $\mathcal{P}' = \bigcup\{\mathcal{P}_{n,i}: n, i \in N\}$. Then $\mathcal{P}'$ is a σ-discrete pair collection and $\bigcup \mathcal{P}'^F = A \setminus B$ is a G_δ-set. Now apply Lemma 2.4, and let $\mathcal{Q}$ be the collection obtained in the lemma. Then $\mathcal{Q}' = \{(H, V \setminus B): (H, V) \in \mathcal{Q}\}$ is the desired pair collection. □

3. Characterizations of a stratifiable space

Definition 3.1. Let $\mathcal{U}$ be an open collection of a space X. A sequence $\langle \mathcal{P}_n \rangle$ of pair covers of X is a *stratifier* of $\mathcal{U}$ if for each $U \in \mathcal{U}$, $\bigcap\{\mathrm{Cl}\,\mathrm{A}(U, \mathcal{P}_n): n \in N\} = X \setminus U$. A sequence $\langle \mathcal{P}_n \rangle$ of pair covers of X is a *stratifier of the space X* if $\langle \mathcal{P}_n \rangle$ is a stratifier of $\mathcal{T}$ where $\mathcal{T}$ is the set of all open sets of X. A stratifier $\langle \mathcal{P}_n \rangle$ of an open collection $\mathcal{U}$ is *σ-discrete* (*point finite*) if each $\mathcal{P}_n$ is σ-discrete (point finite).

The notion of a stratifier is a generalization of a sequence of neighbornets and stratifiable spaces are characterized by stratifiers with various assumptions. In particular, we show that every stratifiable space has a σ-discrete and point finite stratifier. This result and the method of G. Gruenhage of showing that every stratifiable F_σ-metrizable space is M_1([6]) suggests the idea of our new class of stratifiable spaces, called regularly stratifiable spaces, which has a stratifier satisfying certain condition of locally finite type (Section 4). We use a σ-discrete stratifier as a tool to study stratifiable μ-spaces (Section 5).

The proof of the following lemma is straightforward.

Lemma 3.2. *Let $\mathcal{U}$ be an open collection of a space X and $\langle \mathcal{P}_n \rangle$ is a stratifier of $\mathcal{U}$. If $\langle \mathcal{Q}_n \rangle$ is a sequence of pair covers of X and for each $n \in N$, $\mathcal{Q}_n$ weakly refines $\mathcal{P}_n$, then the sequence $\langle \mathcal{Q}_n \rangle$ is also a stratifier of $\mathcal{U}$.*

Theorem 3.3. *The following properties of a space X are equivalent:*

(a) *X is stratifiable;*

(a)′ *X is M_2;*

(b) *X has a stratifier $\langle \mathcal{P}_n \rangle$, where each $\mathcal{P}_n$ is of the form $\{(\{x\}, U_n(x)) : x \in X\}$;*

(b)′ *X has a stratifier $\langle \mathcal{P}_n \rangle$, where each $\mathcal{P}_n$ is of the form $\{(\{x\}, U_n(x)) : x \in X\}$ and furthermore each neighbornet U_n is transitive, i.e. if $x, y \in X$ and $y \in U_n(x)$, then $U_n(y) \subset U_n(x)$;*

(c) *X has a σ-discrete and point finite stratifier;*

(d) *X has a stratifier;*

(e) *X has a base $\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in N\}$ where each $\mathcal{B}_n$ has a stratifier.*

Proof. The equivalence (a) $\Leftrightarrow$ (a)′ is the result of G. Gruenhage [5] and H. Junnila [11]. (a) $\Leftrightarrow$ (b) is due to R. Heath [7] and (a)′ $\Leftrightarrow$ (b)′ is due to J. Nagata [19]. Thus (a), (a)′, (b), and (b)′ are all equivalent. The implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are obvious. To complete the proof we need only show that (b)′ $\Rightarrow$ (c) and (e) $\Rightarrow$ (b).

(b)′ $\Rightarrow$ (c). Let $\langle \mathcal{P}_n \rangle$ be a stratifier of X satisfying the condition of (b)′. Since (b)′ is equivalent to (a), X is stratifiable, hence is paracompact and semi-stratifiable. Using Lemma 2.5 for each pair cover $\mathcal{P}_n$, we have a sequence $\langle \mathcal{Q}_n \rangle$ of σ-discrete and point finite pair covers of X such that each $\mathcal{Q}_n$ weakly refines $\mathcal{P}_n$. Now apply Lemma 3.2. Thus $\langle \mathcal{Q}_n \rangle$ is a σ-discrete and point finite stratifier of X.

(e) $\Rightarrow$ (b). Let $\langle \mathcal{P}_{n,k} \rangle_{k \in N}$ be a stratifier of an open collection $\mathcal{B}_n$. Since each $\mathcal{P}_{n,k}$ is a pair cover of X, for each $x \in X$ and $n, k \in N$, there is $(F_{n,k}(x), U_{n,k}(x)) \in \mathcal{P}_{n,k}$ such that $x \in F_{n,k}(x)$. Define $\mathcal{Q}_{n,k} = \{(\{x\}, U_{n,k}(x)) : x \in X\}$. Then $\langle \mathcal{Q}_{n,k} \rangle_{n,k \in N}$ is a stratifier of X. In fact, let U be an open set of X and $x \in U$. Since $\mathcal{B}$ is a base, there is $n \in N$ and $B \in \mathcal{B}_n$ with $x \in B \subset U$. Since $\langle \mathcal{P}_{n,k} \rangle_{k \in N}$ is a stratifier of $\mathcal{B}_n$, we have $k \in N$ such that $x \notin \mathrm{Cl}\, \mathrm{A}(B, \mathcal{P}_{n,k})$. Observe that $\mathrm{A}(U, \mathcal{Q}_{n,k}) \subset \mathrm{A}(U, \mathcal{P}_{n,k}) \subset \mathrm{A}(B, \mathcal{P}_{n,k})$. Hence $x \notin \mathrm{Cl}\, \mathrm{A}(U, \mathcal{Q}_{n,k})$. The proof is completed. $\square$

Remark 3.4. In the above characterizations of stratifiable spaces, we may assume that for each stratifier $\langle \mathcal{P}_n \rangle$, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in N$. In fact if $\langle \mathcal{P}_n \rangle$ is a stratifier, then $\langle \mathcal{Q}_n \rangle$, where $\mathcal{Q}_n = \mathcal{P}_1 \wedge \mathcal{P}_2 \wedge \cdots \wedge \mathcal{P}_n$, is a stratifier satisfying that each $\mathcal{Q}_{n+1}$ refines $\mathcal{Q}_n$. We define $\mathcal{P} \wedge \mathcal{Q} = \{(F \cap H, U \cap V) : (F, U) \in \mathcal{P}$ and $(H, V) \in \mathcal{Q}\}$. Note that the operation $\mathcal{P} \wedge \mathcal{Q}$ preserves σ-discreteness, point finiteness and some other properties.

4. Regularly stratifiable spaces

Now, we define regularly stratifiable spaces and study properties of regularly stratifiable spaces.

Definition 4.1. Let $\mathcal{P}$ be a pair cover of a space X, $\mathcal{U}$ an open collection of X. $\mathcal{P}$ is *finitely approaching* to $\mathcal{U}$ if for each $U \in \mathcal{U}$, $\mathcal{A}(U, \mathcal{P})$ is locally finite at every point of U. $\mathcal{P}$ is *finitely approaching in* X if $\mathcal{P}$ is finitely approaching to the collection of all open sets of X. A stratifier $\langle \mathcal{P}_n \rangle$ of an open collection $\mathcal{U}$ is *finitely approaching* (to $\mathcal{U}$) if each pair cover $\mathcal{P}_n$ is finitely approaching to $\mathcal{U}$.

The reader should compare the finitely approaching property of pair collections with the almost locally finite property of usual collections in the sense of Lelek [12]. Let X be a space. A collection $\mathcal{A}$ of subsets of X is called almost locally finite if for every open set U of X the collection $\{A \in \mathcal{A} : A \setminus U \neq \emptyset\}$ is locally finite at each point of U. Do not confuse this almost locally finite property with the almost locally finite property in the sense of Itō and the author [10] which will be defined later in this section. Aleksandrov [1] called an open base which is almost locally finite as a *regular base*. It was proved by Arhangel'skiĭ [2] that a T_1-space X is metrizable if and only if X admits a regular base.

Definition 4.2. A space X is *regularly stratifiable* if X has a base $\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in N\}$ where each $\mathcal{B}_n$ has a point finite and finitely approaching stratifier. A space X is *strongly regularly stratifiable* if X has a base $\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in N\}$ where each $\mathcal{B}_n$ has a σ-locally finite, point finite and finitely approaching stratifier.

We do not know whether every regularly stratifiable space is strongly regularly stratifiable.

First, we show that the class of (strongly) regularly stratifiable space is hereditary.

Theorem 4.3. *Let X be a (strongly) regularly stratifiable space and $Y \subset X$. Then Y is (strongly) regularly stratifiable.*

Proof. Let $\mathcal{B} = \bigcup \{\mathcal{B}_n : n \in N\}$ be a base of X where each $\mathcal{B}_n$ has a stratifier $\langle \mathcal{P}_{n,k} \rangle_{k \in N}$ such that each $\mathcal{P}_{n,k}$ is (σ-locally finite), point finite and finitely approaching to $\mathcal{B}_n$. Put $\mathcal{B}'_n = \{B \cap Y : B \in \mathcal{B}_n\}$ and $\mathcal{P}'_{n,k} = \{(F \cap Y, U \cap Y) : (F, U) \in \mathcal{P}_{n,k}\}$. Then $\mathcal{B}' = \bigcup \{\mathcal{B}'_n : n \in N\}$ is a base of Y such that each $\mathcal{B}'_n$ has a (σ-locally finite), point finite and finitely approaching stratifier $\langle \mathcal{P}'_{n,k} \rangle_{k \in N}$. □

We do not know if the class of regularly stratifiable spaces is productive. But we have:

Theorem 4.4. *Let X be a (strongly) regularly stratifiable space and Y a metric space. Then the product $X \times Y$ is (strongly) regularly stratifiable.*

Proof. For each $n \in N$, let $\mathscr{V}_n$ be a locally finite open cover of Y such that $\delta(V) < 1/n$ for each $V \in \mathscr{V}_n$, where $\delta(A)$ is the diameter of a set A with respect to the given metric. Let $\{H_V: V \in \mathscr{V}_n\}$ be a closed cover of Y such that $H_V \subset V$. Put $\mathscr{Q}_n = \{(H_V, V): V \in \mathscr{V}_n\}$. Then the sequence $\langle \mathscr{Q}_n \rangle$ is a locally finite and finitely approaching stratifier of Y.

Let $\mathscr{B} = \bigcup\{\mathscr{B}_n: n \in N\}$ be a base of X such that each $\mathscr{B}_n$ has a (σ-locally finite), point finite and finitely approaching stratifier $\langle \mathscr{P}_{n,k} \rangle_{k \in N}$.

Now define

$$\mathscr{B}'_n = \{B \times W: B \in \mathscr{B}_n \text{ and } W \text{ is an open set of } Y\}.$$

Then $\bigcup\{\mathscr{B}'_n: n \in N\}$ is a base for the space $X \times Y$. Define

$$\mathscr{P}'_{n,k} = \{(F \times H_V, U \times V): (F, U) \in \mathscr{P}_{n,k} \text{ and } V \in \mathscr{V}_k\}.$$

It is easy to check that each $\langle \mathscr{P}'_{n,k} \rangle_{k \in N}$ is a (σ-locally finite), point finite and finitely approaching stratifier of $\mathscr{B}'_n$. $\square$

It follows from Theorem 3.3(e) that every regularly stratifiable space is stratifiable. Moreover we show that every regularly stratifiable space has a σ-almost locally finite base.

Definition 4.5 [10]. Let X be a space, x a point of X and $\mathscr{A}$ a collection of subsets of X. $\mathscr{A}$ is *almost locally finite at* x if:

(*) there exists a neighborhood U of x and a finite collection $\mathscr{B}$ of subsets of X such that for each $A \in \mathscr{A}$, $A \cap U = B \cap V$ for some $B \in \mathscr{B}$ and some (not necessarily open) neighborhood V of x.

$\mathscr{A}$ is *almost locally finite in* X if $\mathscr{A}$ is almost locally finite at every point of X.

Every almost locally finite collection is known to be closure preserving. Hence every space with a σ-almost locally finite base in an M_1-space.

Remark 4.6. It is noted by M. Itō that the condition (*) of the above definition can be replaced by the following simpler condition (**).

(**) there exists a finite collection $\mathscr{B}$ of subsets of X such that for each $A \in \mathscr{A}$, $A = B \cap V$ for some $B \in \mathscr{B}$ and some (not necessarily open) neighborhood V of x.

To prove (*) $\Rightarrow$ (**), put $\mathscr{B}' = \{B \cup (X \setminus U): B \in \mathscr{B}\}$. Use $\mathscr{B}'$ instead of $\mathscr{B}$. Note that

$$\begin{aligned} A &= (A \cap U) \cup (A \setminus U) = (B \cap V \cap U) \cup (A \setminus U) \\ &= (B \cup (X \setminus U)) \cap ((V \cap U) \cup (A \setminus U)) \end{aligned}$$

and $(V \cap U) \cup (A \setminus U)$ is a neighborhood of x.

Lemma 4.7. *Let $\mathscr{V}$ be an open collection of a space X and $\mathscr{P}$ a pair cover of X which is finitely approaching to $\mathscr{V}$. Suppose $\{\mathrm{Cl}\, U: U \in \mathscr{P}^U\}$ is point finite. Then the open* *collection $\mathscr{G}=\{G(V): V\in\mathscr{V}\}$, where $G(V)=X\backslash \mathrm{Cl}\, A(V,\mathscr{P})$, is almost locally finite in X.*

Proof. Let $x\in X$. Take an element (F_0, U_0) of $\mathscr{P}$ such that $x\in F_0\subset U_0$. Put $\mathscr{F}=\{\mathrm{Cl}\, U: U\in\mathscr{P}^U$ and $x\in \mathrm{Cl}\, U\}$, and define $\mathscr{B}=\{X\backslash\bigcup\mathscr{F}': \mathscr{F}'\subset\mathscr{F}\}\cup\{X\}$.

We claim that the neighborhood U_0 of x and the finite collection $\mathscr{B}$ of X guarantees the almost local finiteness of $\mathscr{G}$ at x. We show that for every $V\in\mathscr{V}$, $G(V)\cap U_0$ is of the form $B\cap W$ where $B\in\mathscr{B}$ and W is a neighborhood of x. We distinguish three cases.

Case 1. $x\notin V$. Since $x\in F_0\subset U_0$, $U_0\in\mathscr{A}(V,P)$. So $U_0\cap G(V)=\emptyset$.

Case 2. $x\in G(V)$. Then $G(V)=X\cap G(V)$, $X\in\mathscr{B}$ and $G(V)$ is a neighborhood of x.

Case 3. $x\in V$ and $x\notin G(V)$. Finitely approaching property of $\mathscr{P}$ implies that $\mathscr{A}(V,\mathscr{P})$ is locally finite at x. So there exists a finite subcollection $\mathscr{A}'$ of $\mathscr{A}(V,\mathscr{P})$ such that $x\in\mathrm{Cl}\, U$ for every $U\in\mathscr{A}'$ and $x\notin\mathrm{Cl}\bigcup\{U: U\in\mathscr{A}(V,\mathscr{P})\backslash\mathscr{A}'\}$. Put

$$H=\mathrm{Cl}\bigcup\{U: U\in\mathscr{A}(V,\mathscr{P})\backslash\mathscr{A}'\}.$$

Then

$$B=X\backslash\mathrm{Cl}\bigcup\{U: U\in\mathscr{A}'\}=X\backslash\bigcup\{\mathrm{Cl}\, U: U\in\mathscr{A}'\}\in\mathscr{B}$$

and

$$\begin{aligned} G(V)&=X\backslash\mathrm{Cl}\, A(V,\mathscr{P})=X\backslash\mathrm{Cl}\bigcup\mathscr{A}(V,\mathscr{P})\\ &=X\backslash((\mathrm{Cl}\bigcup\{U: U\in\mathscr{A}'\})\cup(\mathrm{Cl}\bigcup\{U: U\in\mathscr{A}(V,\mathscr{P})\backslash\mathscr{A}'\}))\\ &=B\cap(X\backslash H).\end{aligned}$$

Since $X\backslash H$ is a neighborhood of x, the proof is completed. □

Theorem 4.8. *Every regularly stratifiable space has a σ-almost locally finite base.*

Proof. Let X be a regularly stratifiable space with a base $\mathscr{B}=\bigcup\{\mathscr{B}_n: n\in N\}$ where each $\mathscr{B}_n$ has a point finite and finitely approaching stratifier $\langle\mathscr{P}_{n,k}\rangle_{k\in N}$. Since X is stratifiable, X is normal. So replacing each pair (F, U) by a pair (F, V) such that $F\subset V\subset\mathrm{Cl}\, V\subset U$, we may assume that for each n, k, $\{\mathrm{Cl}\, U: U\in\mathscr{P}^U_{n,k}\}$ is point finite. For each $n, k\in N$, put $\mathscr{G}_{n,k}=\{G(B): B\in\mathscr{B}_n\}$, where $G(B)=X\backslash\mathrm{Cl}\, A(B,\mathscr{P}_{n,k})$. It follows from Lemma 4.7 that each open collection $\mathscr{G}_{n,k}$ is almost locally finite in X. Since each $\langle\mathscr{P}_{n,k}\rangle_{k\in N}$ is a stratifier of $\mathscr{B}_n$, each element of $\mathscr{B}_n$ is a union of some elements of $\bigcup\{\mathscr{G}_{n,k}: k\in N\}$. Hence $\bigcup\{\mathscr{G}_{n,k}: n, k\in N\}$ is a base of X. □

5. Stratifiable μ-spaces

In this section we show that every stratifiable μ-space is strongly regularly stratifiable. Our technique is suggested by that of Gruenhage [6] showing that every stratifiable F_σ-metrizable space is an M_1-space, but we must make some changes.

Definition 5.1. A space is F_σ*-metrizable* if it is a countable union of closed metrizable subspaces. A space is a μ*-space* if it is embedded in a countable product of paracompact F_σ-metrizable spaces.

μ-spaces are defined by Nagami [16] for dimension theoretical purposes. See [14] for the dimension theoretical properties of μ-spaces.

Lemma 5.2. *Let X be a paracompact and perfectly normal space and $X = \bigcup\{X_n : n \in N\}$, where each X_n is a closed subspace of X and $X_1 \subset X_2 \subset \cdots$. Suppose $\langle \mathcal{P}_n \rangle$ is a sequence of σ-discrete pair covers of X and ρ is a continuous pseudometric on X. Let k be a natural number. Then there is a pair cover $\mathcal{Q} = \bigcup\{\mathcal{Q}_n : n \in N\}$ of X such that*

(1) $\bigcup \mathcal{Q}_n^F = X_n \backslash X_{n-1}$ *where* $X_0 = \emptyset$; *and* $\bigcup \mathcal{Q}_n^U \cap X_{n-1} = \emptyset$;

(2) *each* $\mathcal{Q}_n$ *refines* $\mathcal{P}_n$;

(3) *each* $\mathcal{Q}_n$ *is of the form* $\mathcal{Q}_n = \bigcup\{\mathcal{Q}_{n,m} : m \in N\}$ *such that each* $\mathcal{Q}_{n,m}$ *is locally finite and* $\delta(U) < 1/(k+n+m)$ *for* $U \in \mathcal{Q}_{n,m}^U$ *where* $\delta(U)$ *is the diameter of* U *with respect to the given pseudometric* ρ; *and*

(4) $\mathcal{Q}$ *is a* σ*-locally finite and point finite cover of* X.

Proof. For each $n \in N$, by Lemma 2.6, there is a σ-discrete and point finite pair collection $\mathcal{R}_n = \bigcup\{\mathcal{R}_{n,m} : m \in N\}$ such that

(1) $\bigcup \mathcal{R}_n^F = X_n \backslash X_{n-1}$, $\bigcup \mathcal{R}_n^U \cap X_{n-1} = \emptyset$; and

(2) each $\mathcal{R}_{n,m}$ is discrete and each $\mathcal{R}_n$ refines $\mathcal{P}_n$.

Fix $n, m \in N$. Using the paracompactnesss of X, for each $P = (F, U) \in \mathcal{R}_{n,m}$, we have a locally finite pair collection $\mathcal{Q}_P$ of X such that

(1) $\bigcup \mathcal{Q}_P^F = F$, $\bigcup \mathcal{Q}_P^U \subset U$; and

(2) $\delta(V) < 1/(k+n+m)$ for each $V \in \mathcal{Q}_P^U$.

Define $\mathcal{Q}_{n,m} = \bigcup\{\mathcal{Q}_P : P \in \mathcal{R}_{n,m}\}$. Now put $\mathcal{Q}_n = \bigcup\{\mathcal{Q}_{n,m} : m \in N\}$ and $\mathcal{Q} = \bigcup\{\mathcal{Q}_n : n \in N\}$. Then $\mathcal{Q}$ is the desired pair cover. □

Lemma 5.3. *Let X be a stratifiable space, Y a paracompact F_σ-metrizable space, and Z a space such that $X \subset Y \times Z$. Then the open collection $\mathcal{W} = \{(G \times Z) \cap X : G$ is an open set of $Y\}$ of X has a σ-locally finite, point finite and finitely approaching stratifier.*

Proof. Let $Y = \bigcup\{Y_n : n \in N\}$, where each Y_n is a closed metrizable subspace of Y. Since every finite union of closed metrizable subspaces is metrizable, we may assume that $Y_1 \subset Y_2 \subset \cdots$. Let ρY be a metric space on the set Y with a metric ρ such that

(1) The identity transformation ρ of Y to ρY is continuous; and

(2) $\rho \mid Y_n$ is a homeomorphism and $\rho(Y_n)$ is closed for each n. Such a metric exists by Theorem 2.2 of [17].

A continuous pseudometric σ is defined as $\sigma(x_1, x_2) = \rho(\pi(x_1), \pi(x_2))$ where $\pi : Y \times Z \to Y$ is a projection.

Since X is stratifiable, by Theorem 3.3 (c), there is a stratifier $\langle \mathscr{P}_n \rangle$ where each $\mathscr{P}_n$ is σ-discrete. Furthermore by Remark 3.4, we may assume that each $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for $n \in N$.

For each $k \in N$, we define a pair cover $\mathscr{Q}_k$. Apply Lemma 5.2 replacing X_n by $(Y_n \times Z) \cap X$ and ρ by σ. Then there is a pair cover $\mathscr{Q}_k = \bigcup \{\mathscr{Q}_{k,n} : n \in N\}$ of X such that

(1) $\bigcup \mathscr{Q}^F_{k,n} = ((Y_n \setminus Y_{n-1}) \times Z) \cap X$;

(2) each $\mathscr{Q}_{k,n}$ refines $\mathscr{P}_n$;

(3) $\mathscr{Q}_{k,n} = \bigcup \{\mathscr{Q}_{k,n,m} : m \in N\}$ where each $\mathscr{Q}_{k,m,n}$ is locally finite and $\delta(U) < 1/(k+n+m)$ for each $U \in \mathscr{Q}^U_{k,m,n}$;

(4) $\mathscr{Q}_k$ is σ-locally finite and point finite.

We claim that the sequence $\langle \mathscr{Q}_k \rangle$ is the desired stratifier of the open collection $\mathscr{W}$. To see this, it remains to show that $\langle \mathscr{Q}_k \rangle$ is a finitely approaching stratifier.

First we show that each $\mathscr{Q}_k$ is finitely approaching to $\mathscr{W}$. Fix $k \in N$. Let $W = (G \times Z) \cap X$ be an element of $\mathscr{W}$ where G is an open set of Y and let $x \in W$. Since the sequence $\langle \mathscr{P}_n \rangle$ is a stratifier of X and $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for each $n \in N$, there is $n_0 \in N$ such that

$$x \notin \mathrm{Cl}\ \mathrm{A}(W, \bigcup \{\mathscr{P}_n : n > n_0\}).$$

Since each $\mathscr{Q}_{k,n}$ refines $\mathscr{P}_n$ by (2),

$$x \notin \mathrm{Cl}\ \mathrm{A}(W, \bigcup \{\mathscr{Q}_{k,n} : n > n_0, k \in N\}).$$

So it remains to show that $\mathscr{A}(W, \bigcup \{\mathscr{Q}_{k,n} : n = 1, 2, \ldots, n_0, k \in N\}$ is locally finite at x. Note that:

$$\bigcup \bigcup \{\mathscr{Q}^F_{k,n} : n = 1,2 \ldots, n_0, k \in N\} \subset Y_{n_0} \times Z$$

by (1), and

$$(Y_{n_0} \times Z) \cap X \setminus W = ((Y_{n_0} \setminus G) \times Z) \cap X.$$

Hence:

(5) $U \cap ((Y_{n_0} \setminus G) \times Z) \neq \emptyset$ for each element U of $\mathscr{A}(W, \bigcup \{\mathscr{Q}_{k,n} : n = 1, 2, \ldots, n_0, k \in N\})$.

On the other hand, by condition (2) of the metric ρ,

(6) $\sigma(x, ((Y_{n_0} \setminus G) \times Z) \cap X) \geqslant \rho(\pi(x), Y_{n_0} \setminus G) > 0$. Using (3), (5) and (6), it is easy to check that $\mathscr{A}(W, \bigcup \{\mathscr{Q}_{k,n} : n = 1, 2, \ldots, n_0, k \in N\})$ is locally finite at x.

Finally we show that the sequence $\langle \mathscr{Q}_k \rangle$ is a stratifier of $\mathscr{W}$. Fix $k \in N$, $W = (G \times Z) \cap X$ and $x \in W$ of the above proof of the finitely approaching property. Now choose $k' \in N$ such that $1/k' < \sigma(x, (Y_{n_0} \setminus G) \times Z)$. It is easy to check that $x \notin \mathrm{Cl}\ \mathrm{A}(W, \mathscr{Q}_{k'})$. The proof is completed. □

Theorem 5.4. *Every stratifiable μ-space X is strongly regularly stratifiable.*

Proof. Suppose $X \subset \prod_{k=1}^{\infty} Y_k$ where each Y_k is a paracompact F_σ-metrizable space.

Note that every finite product of paracompact F_σ-metrizable spaces is also paracompact F_σ-metrizable. Since $X \subset \prod_{k=1}^{n} Y_k \times \prod_{k=n+1}^{\infty} Y_k$, by Lemma 5.3, the open collection

$$\mathcal{B}_n = \left\{\left(G \times \prod_{k=n+1}^{\infty} Y_k\right) \cap X;\ G \text{ is an open set of } \prod_{k=1}^{n} Y_k\right\}$$

has a σ-locally finite, point finite and finitely approaching stratifier. Since $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in N\}$ is a base of X, X is strongly regularly stratifiable. □

In particular we have the following:

Theorem 5.5. *Every stratifiable F_σ-metrizable space has a σ-locally finite, point finite and finitely approaching stratifier.*

Proof. Let Z be a singleton space. Then X is naturally embedded in $X \times Z$. Apply Lemma 5.3. Then $\mathcal{W}$ is a base of X. □

Corollary 5.6 [6]. *Every stratifiable F_σ-metrizable space is M_1.*

Corollary 5.7 [15]. *Every stratifiable μ-space is M_1.*

6. Dimension theoretical properties

In this section, first we show that J. Nagata's problem is answered positively in the class of regularly stratifiable spaces. Next we show that every strongly regularly stratifiable space is in the class EM_3 which was defined by Oka [21]. The class EM_3 is a subclass of stratifiable spaces and behaves very well with respect to dimension theory. Using these results we show that every strongly regularly stratifiable space is the perfect image of an M_0-space.

Lemma 6.1. *Let V be an open set of a totally normal space X. Suppose $\mathcal{P}$ is a pair cover of X such that $\mathcal{P}$ is finitely approaching to $\{V\}$, and* $\operatorname{Ind} \partial U \leq n$ *for each $(F, U) \in \mathcal{P}$. Define $G = X \setminus \operatorname{Cl} \mathrm{A}(V, \mathcal{P})$. Then* $\operatorname{Ind} \partial G \leq n$.

Proof. By the finitely approaching property of $\mathcal{P}$, $\mathcal{A}(V, \mathcal{P})$ is locally finite at every point of V. Hence

$$\operatorname{Cl} \mathrm{A}(V, \mathcal{P}) = \bigcup\{\operatorname{Cl} U : U \in \mathcal{A}(V, \mathcal{P})\}.$$

So

$$\partial G = \bigcup\{\operatorname{Cl} U : U \in \mathcal{A}(V, \mathcal{P})\} \setminus \bigcup \mathcal{A}(V, \mathcal{P}) \subset \bigcup\{\partial U : U \in \mathcal{A}(V, \mathcal{P})\}.$$

Since $\partial G \subset V$ and $\{\partial U : U \in \mathcal{A}(V, \mathcal{P})\}$ is locally finite at each point of V, $\{\partial G \cap \partial U : U \in \mathcal{A}(V, \mathcal{P})\}$ is a locally finite closed cover of the totally normal space ∂G.

For each $U \in \mathscr{A}(V, \mathscr{P})$, $\operatorname{Ind} \partial U \leqslant n$ implies that $\operatorname{Ind} \partial G \cap \partial U \leqslant n$. Note that if Y is a totally normal space and $\mathscr{F}$ is a locally finite closed covering of Y such that $\operatorname{Ind} F \leqslant n$ for each $F \in \mathscr{F}$, then $\operatorname{Ind} Y \leqslant n$. (See [22, 4.4.10].) Therefore $\operatorname{Ind} \partial G \leqslant n$, which completes the proof. □

Theorem 6.2. *Let X be a regularly stratifiable space. Then* $\operatorname{Ind} X \leqslant n$ *if and only if X has a σ-almost locally finite open base $\mathscr{B}$ such that* $\operatorname{Ind} \partial B \leqslant n-1$ *for each $B \in \mathscr{B}$.*

Proof. The 'if' part of the theorem follows from the result of Nagata [20] that if X has a σ-closure preserving base $\mathscr{B}$ such that $\operatorname{Ind} \partial B \leqslant n-1$ for each $B \in \mathscr{B}$, then $\operatorname{Ind} X \leqslant n$.

Now we show the converse. Let X be a regularly stratifiable space with $\operatorname{Ind} X \leqslant n$. Let $\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in N\}$ be a base of X such that each $\mathscr{B}_n$ has a point finite and finitely approaching stratifier $\langle \mathscr{P}_{n,k} \rangle_{k \in N}$. We proceed same as Theorem 4.8, but we replace each pair (F, U) by a pair (F, V) such that $F \subset V \subset \operatorname{Cl} V \subset U$ and furthermore $\operatorname{Ind} \partial V \leqslant n-1$. This can be done using the fact that $\operatorname{Ind} X \leqslant n$. By Theorem 4.8 and Lemma 6.1, we obtain a σ-almost locally finite base $\bigcup\{\mathscr{G}_{n,k} : n, k \in N\}$ such that $\operatorname{Ind} \partial G \leqslant n-1$ for each $G \in \bigcup\{\mathscr{G}_{n,k} : n, k \in N\}$. □

Corollary 6.3. *Every strongly* 0*-dimensional regularly stratifiable space is an M_0-space, i.e. a space with a σ-closure preserving base consisting of clopen sets.*

Definition 6.4 [21]. Let X be a space. A collection $\mathscr{E}$ of subsets of X is *an encircling net* if for each open set U and $x \in U$, there is a subcollection $\mathscr{F}$ of $\mathscr{E}$ such that $x \in X \setminus \bigcup \mathscr{F} \subset U$ and $\bigcup \mathscr{F}$ is a closed subset of X. EM_3 is the class of stratifiable spaces with a σ-closure preserving encircling net.

S. Oka obtained several dimension theoretical results for the class EM_3 including the coincidence theorem for dim and Ind. In particular he proved the following:

Theorem 6.5 [21]. *The following properties of a space X are equivalent:*

(a) *X is a member of EM_3 (with* $\operatorname{Ind} X \leqslant n$*).*

(b) *X is the image of a stratifiable space X_0 with* $\operatorname{Ind} X_0 = 0$ *under a perfect map (of order not greater than $n+1$).*

Remark 6.6. Observing the proof of S. Oka, we may assume that X_0 of the above theorem is a subspace of the product of X and a metric space.

Theorem 6.7. *Every strongly regularly stratifiable space is a member of EM_3.*

Proof. Let X be a strongly regularly statifiable space with a base $\mathscr{B} = \bigcup\{\mathscr{B}_n : n \in N\}$ where each $\mathscr{B}_n$ has a stratifier $\langle \mathscr{P}_{n,k} \rangle_{k \in N}$ such that each $\mathscr{P}_{n,k}$ is σ-locally finite, point finite and finitely approaching to $\mathscr{B}_n$. Then $\bigcup\{\mathscr{P}^F_{n,k} : n,k \in N\}$ is a σ-locally finite closed collection. We show that $\bigcup\{\mathcal{P}^F_{n,k}: n, k \in N\}$ is an encircling net. Let U be an open set of X and $x \in U$. Since $\mathcal{B}$ is a base, there is $n \in N$ and $B \in \mathcal{B}_n$ such that $x \in B \subset U$. Since $\langle \mathcal{P}_{n,k} \rangle_{k \in N}$ is a stratifier of $\mathcal{B}_n$, there is $k \in N$ such that $x \notin \mathrm{Cl}\, A(B, \mathcal{P}_{n,k})$. The finitely approaching property of $\mathcal{P}_{n,k}$ implies that $\mathcal{F} = \{F \in \mathcal{P}^F_{n,k}: F \backslash B \neq \emptyset\}$ is locally finite at every point of B. Hence $\bigcup \mathcal{F}$ is closed and $x \in X \backslash \bigcup \mathcal{F} \subset B \subset U$. □

It is known [8] that if every M_1-space is the image of an M_0-space under a perfect mapping, then every stratifiable space is an M_1-space. We have the following partial result:

Theorem 6.8. *Every strongly regularly stratifiable space (with* $\mathrm{Ind}\, X \leq n$*) is the image of an* M_0*-space under a perfect mapping (of order not greater than* $n+1$*).*

Proof. Let X be a strongly regularly stratifiable space (with $\mathrm{Ind}\, X \leq n$). By Theorem 6.7, X is a member of EM_3. Using Theorem 6.5, X is the image of a stratifiable space X_0 with $\mathrm{Ind}\, X_0 = 0$ under a perfect map (of order not greater than $n+1$). By Remark 6.6, we may assume that X_0 is a subspace of the product of X and a metric space. It follows from Theorem 4.3 and Theorem 4.4 that this X_0 is strongly regularly stratifiable. Hence by Corollary 6.3, X_0 is an M_0-space. The proof is completed. □

Corollary 6.9 [15]. *Every stratifiable* μ*-space (with* $\mathrm{Ind}\, X \leq n$*) is the perfect image of an* M_0*-space under a perfect mapping (of order not greater than* $n+1$*).*

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